・不定積分 ―― 微分の逆演算,

・定積分 ―― 面積の計算.

§不定積分

微分の演算 $\frac{d}{dx}$ を関数 $f(x)$ に作用させると,

$$\frac{d}{dx} f(x) = f'(x)$$

微分の逆演算 $\left(\frac{d}{dx}\right)^{-1}$ を関数 $f(x)$ に作用させると,

$$\left(\frac{d}{dx}\right)^{-1} f(x) = F(x).$$

と書ける. $F(x)$ は方程式

$$\frac{d}{dx} F(x) = f(x) \qquad (☆)$$

をみたす関数と定義する.

ただちに分るように, $F(x)$ が方程式 (☆) をみたすとき

$F(x)+C$ もまた (☆) の解となる. (C: 任意定数)

よって, 必らず

$$\left(\frac{d}{dx}\right)^{-1} f(x) = F(x) + C$$

となる. 今, 微分の逆演算 $\left(\frac{d}{dx}\right)^{-1}$ を $\int dx$ と書くとすると

$$\int f(x)\, dx = F(x) + C$$

となる. ここで

$\int f(x)\,dx$ を 不定積分, $f(x)$ を 被積分関数,

C を 積分定数

と呼ぶ. 方程式 (☆) が成り立つとき $F(x)$ を $f(x)$ の原始関数 と呼ぶ.

Def

$$\boxed{\int f(x)\, dx = F(x) + C \quad \overset{\text{def}}{\Longleftrightarrow} \quad \frac{d}{dx} F(x) = f(x)}$$

$\boxed{\text{定理}}$（不定積分の性質）

(1) $\dfrac{d}{dx}\displaystyle\int f(x)\,dx = f(x)$

(2) $\displaystyle\int F'(x)\,dx = F(x) + C$

(3) $\displaystyle\int \alpha f(x)\,dx = \alpha \int f(x)\,dx$　（α：定数）

(4) $\displaystyle\int (f+g)\,dx = \int f\,dx + \int g\,dx$

（証明）$\displaystyle\int f(x)\,dx = F(x) + C \iff \frac{dF}{dx} = f(x)$ とする。

(1) （左辺）$= \dfrac{d}{dx}\displaystyle\int f(x)\,dx = \frac{d}{dx}\{F(x)+C\} = \frac{dF}{dx} = f(x)$ //

(2) ~~$f(x) = F'(x)$ とおく $\int f(x)\,dx = F(x)+C$ ならば $\frac{dF}{dx} = \frac{dF}{dx}$.~~

$\displaystyle\int F'(x)\,dx = G(x) + C$. とする

$F'(x) = \dfrac{d}{dx}(G(x)+C)$

$\dfrac{dF}{dx} = \dfrac{dG}{dx}$　よって $G = F + C$ //

(3). $\displaystyle\int \alpha f(x)\,dx = F(x) + C \iff \frac{dF}{dx} = \alpha f(x)$.

$\alpha\displaystyle\int f(x)\,dx = G(x) + D \iff \frac{d}{dx}\left(\frac{G(x)+D}{\alpha}\right) = f(x) \to \frac{d}{dx}(G+D) = \alpha f(x) \to \frac{dG}{dx} = f(x)$.

(4) $\displaystyle\int f\,dx = F + C \iff \frac{dF}{dx} = f$, $\displaystyle\int g\,dx = G + C \iff \frac{dG}{dx} = g$.

$\displaystyle\int (f+g)\,dx = H + C \iff \frac{dH}{dx} = f + g = \frac{dF}{dx} + \frac{dG}{dx} = \frac{d}{dx}(F+G) \iff H = F + G + C$

不定積分の基本的な計算.

微分の基本的な公式の逆演算を行う.

(1) $\dfrac{d}{dx}x = 1 \iff \displaystyle\int dx = x + C$.

(2) $\dfrac{d}{dx}x^{n+1} = (n+1)x^n \iff \displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$

$(n \neq -1)$

~~$\dfrac{d}{dx}\log x = \dfrac{1}{x} \ (x>0)$~~

(3) $\dfrac{d}{dx}x^{\alpha+1} = (\alpha+1)x^{\alpha} \iff \displaystyle\int x^{\alpha}\,dx = \frac{x^{\alpha+1}}{\alpha+1} + C \quad \begin{pmatrix}\alpha \in \mathbb{R} \\ \alpha \neq -1\end{pmatrix}$

$(\alpha \in \mathbb{R},\ \alpha \neq -1)$

(4) $\left.\begin{cases} \dfrac{d}{dx}\log x = \dfrac{1}{x} & (x>0) \\ \dfrac{d}{dx}\log(-x) = \dfrac{1}{x} & (x<0) \end{cases}\right\} \iff \displaystyle\int \frac{1}{x}\,dx = \log|x| + C$

(5) $\dfrac{d}{dx}e^x = e^x \iff \displaystyle\int e^x\,dx = e^x + C$

(6) $\dfrac{d}{dx}a^x = (\log a)\,a^x \iff \displaystyle\int a^x\,dx = \frac{a^x}{\log a} + C$

$(a>0,\ a \neq 1) \qquad (a>0,\ a \neq 1)$

(7) $\dfrac{d}{dx}\cos x = -\sin x \iff \displaystyle\int \sin x\,dx = -\cos x + C$

(8) $\dfrac{d}{dx}\sin x = \cos x \iff \displaystyle\int \cos x\,dx = \sin x + C$

(9) $\dfrac{d}{dx}\tan x = \dfrac{1}{\cos^2 x} \iff \displaystyle\int \frac{dx}{\cos^2 x} = \tan x + C$

(10) $\dfrac{d}{dx}\arcsin x = \dfrac{1}{\sqrt{1-x^2}} \ (|x|<1) \iff \displaystyle\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C \quad (|x| < 1)$

(11) $\dfrac{d}{dx}\arctan x = \dfrac{1}{1+x^2} \iff \displaystyle\int \frac{dx}{1+x^2} = \arctan x + C$

(12) $\frac{d}{dx}\cancel{\text{arcsinh}\,x}\ \cosh x = \cancel{\cosh x}\ \sinh x \iff \int \sinh x = \cosh x + C$

(13) $\frac{d}{dx}\cancel{\text{arccosh}\,x}\ \sinh x = \cosh x \iff \int \cosh x = \sinh x + C$

(14) $\frac{d}{dx}\cancel{\text{arctan}\,x}\ \tanh x = \frac{1}{\cosh^2 x} \iff \int \frac{dx}{\cosh^2 x} = \tanh x + C$

(15) $\frac{d}{dx}\operatorname{arcsinh} x = \frac{1}{\sqrt{1+x^2}} \iff \int \frac{dx}{\sqrt{1+x^2}} = \cancel{\sinh x}\ \operatorname{arcsinh} x + C$

(16) $\frac{d}{dx}\operatorname{arccosh} x = \frac{1}{\sqrt{x^2-1}} \quad (|x|>1) \iff \int \frac{dx}{\sqrt{x^2-1}} = \operatorname{arccosh} x + C$ $(|x|>1)$

(17) $\frac{d}{dx}\operatorname{arctanh} x = \frac{1}{1-x^2} \quad (x \neq 1) \iff \int \frac{dx}{1-x^2} = \operatorname{arctanh} x + C$

例

(1) $\int x^8\,dx = \frac{1}{9}x^9 + C$

(2) $\int x^{\frac{1}{4}}\,dx = \frac{1}{\frac{1}{4}+1} x^{\frac{1}{4}+1} + C = \frac{4}{5}x^{\frac{5}{4}} + C$

(3) $\int \frac{dx}{x^3} = \int x^{-3}\,dx = \frac{1}{-3+1}x^{-3+1} + C = -\frac{1}{2}x^{-2} + C$

(4) $\int (x^3 - x^2 + 3x - 2)\,dx = \int x^3\,dx - \int x^2\,dx + 3\int \cancel{x+}x - 2\int dx$

$= \frac{1}{4}x^4 - \frac{1}{3}x^3 + 3\times\frac{1}{2}x^2 - 2\times 1 + C = \frac{1}{4}x^4 - \frac{1}{3}x^3 + \frac{3}{2}x^2 + C$

置換積分法

定理 (置換積分)

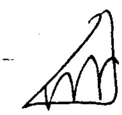
$$\int f(x)\,dx = \int f(\varphi(t))\,\varphi'(t)\,dt = \int f(\varphi(t))\,\frac{dx}{dt}\,dt.$$

ただし $x=\varphi(t)$

~~実際は~~ または.

$$\int f(\psi(x))\,\psi'(x)\,dx = \int f(u)\,du = \underset{+C}{F(u)} = F(\psi(x)) + C$$

ただし $\begin{cases} u=\psi(x) \\ F(u)=\int f(u)\,du \end{cases}$



$F(x) = \int f(x)\,dx$ とおく. $\Longleftrightarrow$ $\frac{dF}{dx} = f(x)$

$x=\varphi(t)$

$$\frac{d}{dt}F(x) = \frac{d}{dt}F(\varphi(t)) = F(x)\frac{dy}{dt} = F'(x)\frac{d\varphi}{dt} = \frac{dF}{dx}\varphi'(t) = f(x)\varphi'(t)$$
$$= f(\varphi(t))\,\varphi'(t).$$

$$\int \frac{d}{dt}F(\varphi(t)) = \int f(\varphi(t))\,\varphi'(t)\,dt.$$

$$\int f(\varphi)\,\varphi'(t)\,dt = F(x) = F(\varphi(t)) + C$$

$$f(\varphi(t))\,\varphi'(t) = \frac{dF}{dt}$$

(証明)

$F(x) = \int f(x)\,dx$ とすると

$F(x) = F(\varphi(t)) = \int f(\varphi(t))\,\varphi'(t)\,dt$ が成り立たなければならない.

定義より すなわち

$$\frac{d}{dt}F(\varphi(t)) = f(\varphi(t))\,\varphi'(t)$$

が成り立つことを示す.

$\frac{dF}{dx} = f(x)$

$$\text{左辺} = \frac{d}{dt}F(\varphi(t)) = F'(\varphi(t))\frac{d\varphi(t)}{dt} = F'(x)\,\varphi'(t) = f(x)\,\varphi'(t)$$
$$= f(\varphi(t))\,\varphi'(t)$$
$$= \text{右辺}.$$

となるので証明終了.

例

(1) $\int (ax+b)^n dx$　　$t=ax+b$ とおくと.

$t=ax+b$ を x で微分して

$\frac{dt}{dx}=a \to \frac{dx}{dt}=\frac{1}{\frac{dt}{dx}}=\frac{1}{a}$

$=\int t^n \frac{dx}{dt}dt = \frac{1}{a}\int t^n dt$

$n \neq -1$ のとき　$=\frac{1}{a}\frac{t^{n+1}}{n+1}+C=\frac{(ax+b)^{n+1}}{a(n+1)}+C$

$n=-1$ のとき

$I=\frac{1}{a}\int t^{-1}dt = \frac{1}{a}\log|t|+C=\frac{1}{a}\log|ax+b|+C$

よって　$\int (ax+b)^n dx = \begin{cases} \frac{(ax+b)^{n+1}}{a(n+1)}+C & (n\neq -1) \\ \frac{1}{a}\log|ax+b|+C & (n=-1) \end{cases}$

$I=\int (ax+b)^n dx = \frac{1}{a}\int (ax+b)^n (ax+b)' dx$

$=\frac{1}{a}\frac{1}{n+1}(ax+b)^{n+1}+C$

(2) $I=\int \cos(ax+b)\,dx$

$ax+b=t$ とおく.

x で微分して.

$a=\frac{dt}{dx}$

よって. $\frac{dx}{dt}=\frac{1}{\frac{dt}{dx}}=\frac{1}{a}$.

$I=\int \cos t \frac{dx}{dt}dt = \frac{1}{a}\int \cos t\, dt$

$=\frac{1}{a}\sin t + C = \frac{1}{a}\sin(ax+b)+C$

$I=\int \cos(ax+b)dx = \frac{1}{a}\int \cos(ax+b)(ax+b)'dx$

$=\frac{1}{a}\sin(ax+b)$

(3) $I=\int \frac{dx}{\sqrt{x^2+a^2}} = \frac{1}{\sqrt{a^2}}\int \frac{dx}{\sqrt{1+\left(\frac{x}{a}\right)^2}} = \frac{1}{\sqrt{a^2}}\int \frac{1}{\sqrt{1+t^2}}\left|\frac{dx}{dt}\right|dt$

$(a>0)$

$\frac{x}{a}=t$ とおく.

x で微分して

$\frac{1}{a}=\frac{dt}{dx} \to \frac{dx}{dt}=\frac{1}{\frac{dt}{dx}}=a.$

$=\frac{\cancel{a}}{\cancel{\sqrt{a^2}}}\int \frac{dt}{\sqrt{1+t^2}} = \operatorname{arcsinh} t + C$

$=\operatorname{arcsinh}\left(\frac{x}{a}\right)+C$

$\operatorname{arcsinh} x = \log\left(x+\sqrt{x^2+1}\right)$

$=\log\left(\frac{x}{a}+\sqrt{\frac{x^2}{a^2}+1}\right)+C$

$=\log\frac{1}{a}\left(x+\sqrt{x^2+a^2}\right)+C$

$=\log\left|x+\sqrt{x^2+a^2}\right|+C$

$I=\frac{1}{a}\int \frac{dx}{\sqrt{1+(x/a)^2}} = \frac{\cancel{a}}{\cancel{a}}\int \frac{(x/a)'}{\sqrt{1+(x/a)^2}}dx$

$=\operatorname{arcsinh}\left(\frac{x}{a}\right)+C$

部分積分法

$\boxed{\text{定理}}$ （部分積分）

$$\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx$$

証明） fg を微分すると

$$(fg)' = f'g + fg'$$

両辺を ~~微分~~ 積分して

$$\int (fg)'\,dx = \int f'g\,dx + \int fg'\,dx.$$

$$fg + C' = \int f'g\,dx + \int fg'\,dx.$$

$$\Rightarrow \int fg'\,dx = \cancel{\int f'g\,dx +}\ fg - \int f'g\,dx - C'$$

$$= fg - \int f'g\,dx$$

不定積分の定数の不定性に繰り込んだ

$\boxed{\text{例}}$

(1) $I = \int \log x\,dx = \int \log x\,(x)'\,dx = x\log x - \int (\log x)'\,x\,dx$

$= x\log x - \int \frac{1}{\cancel{x}}\cancel{x}\,dx = x\log x - \int dx = x\log x - x + C$

(2) $I = \int x\sin x\,dx = \int x(-\cos x)'\,dx = -x\cos x - \int (x)'(-\cos x)\,dx$

$= -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C'$

部分分数分解

有理関数

$$f(x) = \frac{a_0x^n + a_1x^{n-1} + \cdots + a_n}{b_0x^m + b_1x^{m-1} + \cdots + b_m} \qquad n \in \mathbb{N},\ m \in \mathbb{N}$$

の積分 $\int f(x)dx$ をもとめる. 任意の有理式は積分可能である.

分子の次数が分母の次数より大きいとき ($n \geq m$) を考える.

分子を分母で割り

$f(x) =$ 多項式 + 有理式

の形に変形する このとき 有理式の次数は 分子の方が小さくなる.

$\boxed{\text{例}}$

$$f(x) = \frac{x^3+4x^2+2x+1}{x^2+3}$$

$$\begin{array}{r} \overbrace{x+4}^{\text{商}} \\ x^2+3\ \big)\ \overline{x^3+4x^2+2x+1} \\ \underline{x^3 \qquad\ +3x \quad} \\ 4x^2-x+1 \\ \underline{4x^2 \qquad +12} \\ \underbrace{-x-11}_{\text{余り}} \end{array}$$

$$f(x) = \frac{(x+4)(x^2+3) + (-x-11)}{x^2+3}$$

$$= x+4 + \frac{-x-11}{x^2+3} = x+4 - \frac{x-11}{x^2+3}$$

1次

2次

$$\rightarrow \int f(x)dx = \frac{x^2}{2} + 4x - \int \frac{x-11}{x^2+3}dx$$

よって ~~例~~ 分子の次数が 分母の次数より小さいものを考える.

$n < m$ である任意の有理式は

$$\frac{1}{(x+a)^m},\ \frac{Ax+B}{((x-a)^2+b^2)^m} = \cancel{\frac{Ax+B}{(x^2-2ax+a^2+b^2)^n}}$$

~~x^2+bx+c~~

2次式

の形の有理式の和 ~~言~~ 分解できる.

$\boxed{\text{例}}$

$$\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} = \frac{A(x^2-x+1) + (x+1)(Bx+C)}{(x+1)(x^2-x+1)}$$

$$= \frac{A(x^2-x+1) + Bx^2+(B+C)x+C}{\quad\prime\prime\quad} = \frac{(A+B)x^2 + (B+C-A)x + A + C}{\quad\prime\prime\quad}$$

$$\begin{cases} A+B=0 \\ B+C-A=0 \\ \cdots \end{cases} \qquad \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix}1&1&0\\-1&1&1\\1&0&1\end{bmatrix}\begin{bmatrix}A\\B\\C\end{bmatrix}=\begin{bmatrix}0\\0\\1\end{bmatrix} \to \left[\begin{array}{ccc|c}1&1&0&0\\0&1&1&0\\0&0&1&1\end{array}\right] \xrightarrow{\times 1} \left[\begin{array}{ccc|c}1&1&0&0\\0&2&1&0\\0&0&1&1\end{array}\right] \xrightarrow{\times\frac{1}{2}} \left[\begin{array}{ccc|c}1&1&0&0\\0&1&\frac{1}{2}&0\\0&0&1&1\end{array}\right]$$

$$\to \left[\begin{array}{ccc|c}1&0&-\frac{1}{2}&0\\0&1&\frac{1}{2}&0\\0&0&1&1\end{array}\right] \to \left[\begin{array}{ccc|c}1&0&0&\frac{1}{2}\\0&1&0&-\frac{1}{2}\\0&0&1&1\end{array}\right]$$

$A=,$

$$\left[\begin{array}{ccc|c}1&1&0&0\\-1&1&1&0\\1&0&1&1\end{array}\right] \to \left[\begin{array}{ccc|c}1&1&0&0\\0&2&1&0\\0&-1&1&1\end{array}\right] \to \left[\begin{array}{ccc|c}1&1&0&0\\0&-1&1&1\\0&2&1&0\end{array}\right] \to \left[\begin{array}{ccc|c}1&0&1&1\\0&-1&1&1\\0&0&3&2\end{array}\right] \begin{matrix}\times -1\\ \times\frac{1}{3}\end{matrix} \to \left[\begin{array}{ccc|c}1&0&1&1\\0&1&-1&-1\\0&0&1&\frac{2}{3}\end{array}\right]$$

$$\to \left[\begin{array}{ccc|c}1&0&0&\frac{1}{3}\\0&1&0&-\frac{1}{3}\\0&0&1&\frac{2}{3}\end{array}\right] \qquad A=\frac{1}{3},\ B=-\frac{1}{3},\ C=\frac{2}{3}$$

よって $f(x)=\dfrac{1}{x^3+1}=\dfrac{\frac{1}{3}}{x+1}+\dfrac{-\frac{1}{3}x+\frac{2}{3}}{x^2-x+1}=\dfrac{1}{3}\left\{\dfrac{1}{x+1}-\dfrac{x-2}{x^2-x+1}\right\}$

$=\dfrac{1}{3}\left\{\dfrac{1}{x+1}-\dfrac{x-2}{(x-\frac{1}{2})^2+\frac{3}{4}}\right\}$ $\quad \int f(x)dx=\dfrac{1}{3}\int\dfrac{dx}{x+1}-\dfrac{1}{3}\int\dfrac{x-2}{(x-\frac{1}{2})^2+3/4}dx$

$\boxed{例}$ $\dfrac{x-11}{x^3+1}=\dfrac{A}{(x+1)}+\dfrac{Bx+C}{(x^2-x+1)}=\dfrac{(A+B)x^2+(-A+B+C)x+(A+C)}{(x+1)(x^2-x+1)}$

$$\begin{cases}A+B=0\\-A+B+C=1\\A+C=-11\end{cases} \qquad \left[\begin{array}{ccc|c}1&1&0&0\\-1&1&1&1\\1&0&1&-11\end{array}\right] \to \left[\begin{array}{ccc|c}1&0&0&-4\\0&1&0&4\\0&0&1&7\end{array}\right]$$

$\underline{A=-4,\ B=4,\ C=7.}$

$\dfrac{x-11}{x^3+1}=\dfrac{-4}{x+1}+\dfrac{4x+7}{x^2-x+1}.$

$$\left[\begin{array}{ccc|c}1&1&0&0\\-1&1&1&1\\1&0&1&-11\end{array}\right] \to \left[\begin{array}{ccc|c}1&1&0&0\\0&2&1&1\\0&-1&1&-11\end{array}\right]\times\frac{1}{2} \to \left[\begin{array}{ccc|c}1&1&0&0\\0&1&\frac{1}{2}&\frac{1}{2}\\0&-1&1&-11\end{array}\right]$$

$$\to \left[\begin{array}{ccc|c}1&0&-\frac{1}{2}&-\frac{1}{2}\\0&1&\frac{1}{2}&\frac{1}{2}\\0&0&\frac{3}{2}&-\frac{21}{2}\end{array}\right]\times\frac{2}{3} \to \left[\begin{array}{ccc|c}1&0&-\frac{1}{2}&-\frac{1}{2}\\0&1&\frac{1}{2}&\frac{1}{2}\\0&0&1&-\frac{21}{2}\cdot\frac{2}{3}=-7\end{array}\right] \to \left[\begin{array}{ccc|c}1&0&0&-4\\0&1&0&4\\0&0&1&-7\end{array}\right]$$

$$f(x) = \frac{1}{x+a}$$

$$\int f(x)\,dx = \int \frac{dx}{x+a} = \log|x+a| + C.$$

$$f(x) = \frac{1}{(x+a)^m} \quad (m \geqq 2)$$

$$\int f(x)\,dx = \int \frac{dx}{(x+a)^m} = \frac{-1}{m-1}\frac{1}{(x+a)^{m-1}} + C.$$

$$\frac{Ax+B}{((x-a)^2+b^2)^m} = \frac{A(x-a)+aA+B}{((x-a)^2+b^2)^m} = A\frac{x-a}{((x-a)^2+b^2)^m} + (aA+B)\frac{1}{((x-a)^2+b^2)^m}$$

$$= \frac{A}{2}\boxed{\frac{2(x-a)}{((x-a)^2+b^2)^m}} + (aA+B)\boxed{\frac{1}{((x-a)^2+b^2)^m}}$$

$$\begin{cases} I_m = \displaystyle\int \frac{2(x-a)}{((x-a)^2+b^2)^m}\,dx \\ J_m = \displaystyle\int \frac{dx}{((x-a)^2+b^2)^m} \end{cases}$$ とおく.

$$I_m = \int \frac{2(x-a)}{((x-a)^2+b^2)^m}\,dx = \int \frac{((x-a)^2+b^2)'}{((x-a)^2+b^2)^m}\,dx$$

$$= \begin{cases} \dfrac{-1}{m-1}\dfrac{1}{((x-a)^2+b^2)^{m-1}} + C \\ \log|(x-a)^2+b^2| + C \end{cases}$$

$$J_1 = \int \frac{dx}{(x-a)^2+b^2} = \frac{1}{b^2}\int \frac{dx}{1+\left(\frac{x-a}{b}\right)^2} = \frac{b}{b^2}\int \frac{\left(\frac{x-a}{b}\right)'}{1+\left(\frac{x-a}{b}\right)^2}\,dx$$

$$= \frac{1}{b}\arctan\left(\frac{x-a}{b}\right) + C.$$

$\boxed{例1}$ $f(x) = \dfrac{1}{x^3+1} = \dfrac{1}{3}\left\{ \dfrac{1}{x+1} - \dfrac{x-2}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}} \right\}$

$$= \frac{1}{3}\left(\frac{1}{x+1} - \frac{x-2}{x^2-x+1} \right)$$

$$\int f(x)\,dx = \frac{1}{3}\boxed{\int \frac{dx}{x+1}} - \frac{1}{3}\boxed{\int \frac{x-2}{x^2-x+1}\,dx}$$

$$\int \frac{dx}{x+1} = \log|x+1| + C.$$

$$\int \frac{x-2}{x^2-x+1}\,dx = \int \frac{x-2}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\,dx = \int \frac{\left(x-\frac{1}{2}\right)+\overbrace{\left(\frac{1}{2}-2\right)}^{-\frac{3}{2}}}{\left(x-\frac{1}{2}\right)^2+3/4}\,dx$$

$$= \int \frac{x-\frac{1}{2}}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\,dx - \frac{3}{2}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2+3/4}$$

$$I = \int \frac{x-\frac{1}{2}}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\,dx = \int \frac{1}{2}\,\frac{\left(\left(x-\frac{1}{2}\right)^2+\frac{3}{4}\right)'}{\left(x-\frac{1}{2}\right)^2+3/4}\,dx = \frac{1}{2}\log\left|\left(x-\frac{1}{2}\right)^2+3/4\right| + C$$

$$J = \int \frac{dx}{(x-1/2)^2+3/4} = \frac{4}{3}\int \frac{dx}{1+\frac{(x-1/2)^2}{3/4}} = \frac{4}{3}\int \frac{dx}{1+\left(\frac{x-1/2}{\sqrt{3}/2}\right)^2}$$

$$= \frac{4}{3}\int \frac{dx}{1+\left(\frac{2x-1}{\sqrt{3}}\right)^2} = \frac{4}{3}\int \frac{\frac{\sqrt{3}}{2}\left(\frac{2x-1}{\sqrt{3}}\right)'}{1+\left(\frac{2x-1}{\sqrt{3}}\right)^2}\,dx$$

$$= \frac{\cancel{4}^{2}}{3}\times\frac{\sqrt{3}}{\cancel{2}}\int \frac{1}{1+\left(\frac{2x-1}{\sqrt{3}}\right)^2}\left(\frac{2x-1}{\sqrt{3}}\right)'dx$$

$$= \frac{\sqrt{3}}{3}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

$$\int f(x)\,dx = \frac{1}{3}\log|x+1| - \frac{1}{3}\left\{\frac{1}{2}\log\left|\left(x-\frac{1}{2}\right)^2+3/4\right| - \frac{\cancel{3}}{2}\times\frac{\sqrt{3}}{\cancel{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right)\right\} + C'$$

$$= \frac{1}{3}\log|x+1| - \frac{1}{6}\log\left|\left(x-\frac{1}{2}\right)^2+3/4\right| + \frac{3\sqrt{3}}{2}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

$$= \frac{1}{6}\log\left|\frac{(x+1)^2}{\left(x-\frac{1}{2}\right)^2+3/4}\right| + \frac{3\sqrt{3}}{2}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

$$= \frac{1}{6}\log\left|\frac{x^2+2x+1}{x^2-x+1}\right| + \frac{3\sqrt{3}}{2}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C \quad //$$

$$J_m = \int \frac{dx}{((x-a)^2+b^2)^m} \quad (m \geq 2)$$

$$J_m = \int \frac{dx}{\left(b^2\left\{1+\left(\frac{x-a}{b}\right)^2\right\}\right)^m} = \int \frac{1}{b^{2m}} \frac{dx}{\left(1+\left(\frac{x-a}{b}\right)^2\right)^m} \qquad t = \frac{x-a}{b}, \quad \frac{dt}{dx} = \frac{1}{b}$$

$$= \frac{1}{b^{2m}} \int \frac{1}{(1+t^2)^m} \frac{dx}{dt}\, dt = \frac{b}{b^{2m}} \int \frac{dt}{(1+t^2)^m}$$

$$= \frac{1}{b^{2m-1}} \int \frac{dt}{(1+t^2)^m}$$

$$J_m = \frac{1}{b^{2m-1}} \int \frac{dt}{(1+t^2)^m}$$

$$= \frac{1}{b^{2m-1}} \int \frac{(1+t^2)-t^2}{(1+t^2)^m}\, dt = \frac{1}{b^{2m-1}} \int \frac{dt}{(1+t^2)^{m-1}} - \frac{1}{b^{2m-1}} \int \frac{t^2}{(1+t^2)^m}\, dt$$

$$= \cancel{\frac{b^2}{b^{2(m-1)-1}} \int \frac{}{(1+t^2)}}$$

$$\int \frac{t^2}{(1+t^2)^m}\, dt = \cancel{\frac{1}{2}\int \frac{t(1+t^2)'}{(1+t^2)^m}\, dt} = \cancel{\frac{1}{2(m-1)} \int t \left(\frac{1}{(1+t^2)^{m-1}}\right)'}$$

$$= \frac{-1}{2(m-1)} \int t \left(\frac{1}{(1+t^2)^{m-1}}\right)' dt$$

$$= \frac{-1}{2(m-1)} \frac{t}{(1+t^2)^{m-1}} + \frac{1}{2(m-1)} \int \frac{dt}{(1+t^2)^{m-1}}$$

$$J_m = \frac{1}{b^{2m-1}} \int \frac{dt}{(1+t^2)^{m-1}} - \frac{1}{b^{2m-1}} \left\{ \frac{-1}{2(m-1)} \frac{t}{(1+t^2)^{m-1}} + \frac{1}{2(m-1)} \int \frac{dt}{(1+t^2)^{m-1}} \right\}$$

$$= \frac{1}{b^{2m-1}} \left\{ 1 - \frac{1}{2(m-1)} \right\} \int \frac{dt}{(1+t^2)^{m-1}} + \frac{1}{2(m-1)\, b^{2m-1}} \frac{t}{(1+t^2)^{m-1}}$$

$$= \frac{\overset{2m-2-1}{2(m-1)-1}}{2(m-1)} \frac{1}{b^{2m-1}} \int \frac{dt}{(1+t^2)^{m-1}} + \frac{1}{2(m-1)\, b^{2m-1}} \frac{t}{(1+t^2)^{m-1}}$$

$$= \frac{2m-3}{2m-2} \cdot \frac{b^2}{b^{2(m-1)-1}} \int \frac{dt}{(1+t^2)^{m-1}} + \frac{1}{2(m-1)\, b^{2m-1}} \frac{t}{(1+t^2)^{m-1}}$$

$$= \frac{2m-3}{2m-2} b^2 J_{m-1} + \frac{1}{2(m-1)\, b^{2m-1}} \frac{t}{(1+t^2)^{m-1}}$$

$$J_m = \left(\frac{2m-3}{2m-2}\right) b^2 J_{m-1} + \frac{1}{2(m-1)\, b^{2m-1}} \frac{\frac{x-a}{b}}{\left(1+\left(\frac{x-a}{b}\right)^2\right)^{m-1}} \qquad \frac{1}{\cancel{b^{2m-1}}\,\cancel{b}} \quad \frac{1}{b^{2m-2}}$$

$$J_m = \frac{(2m-3)}{(2m-2)b^2} J_{m-1} + \frac{1}{2(m-1)\, b^2} \frac{(x-a)}{((x-a)^2+b^2)^{m-1}} \quad //$$

$\boxed{\text{例4}}$

$$f(x) = \frac{x+1}{x^3+x^2+2x},$$

$x^3+x^2-2x = x(x-1)(x+2)$ より

$$f(x) = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2} \quad \text{とおく.}$$

$$f(x) = \frac{(A+B+C)x^2 + (A+2B-C)x - 2A}{x(x-1)(x+2)}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$A = \frac{1}{\Delta} \begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & 0 & 0 \end{vmatrix} \qquad \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ -2 & 0 & 0 \end{vmatrix} = -2 \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -2(-1-2) = 6$$

$$= \frac{1}{6} \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = \frac{-3}{6} = -\frac{1}{2}$$

$$B = \frac{1}{6} \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ -2 & 1 & 0 \end{vmatrix} = -\frac{1}{6} \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & -2 \\ -2 & 1 & 2 \end{vmatrix} = \frac{2+2}{6} = \frac{2}{3}$$

$$C = \frac{1}{6} \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -2 & 0 & 1 \end{vmatrix} = \frac{1}{6} \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ -2 & 2 & 1 \end{vmatrix} = \frac{1}{6} \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -\frac{1}{6}$$

$$A = -\frac{1}{2},\ B = \frac{2}{3},\ C = -\frac{1}{6}$$

$$f(x) = -\frac{1}{2x} + \frac{2}{3(x-1)} - \frac{1}{6(x+2)}$$

$$\int f(x)\,dx = -\frac{1}{2}\int \frac{dx}{x} + \frac{2}{3}\int \frac{dx}{x-1} - \frac{1}{6}\int \frac{dx}{x+2}$$

$$= -\frac{1}{2}\log|x| + \frac{2}{3}\log|x-1| - \frac{1}{6}\log|x+2| + C'$$

$$= \frac{1}{6}\left\{ -3\log|x| + 4\log|x-1| - \log|x+2| + C' \right\}$$

$$= \frac{1}{6}\log\left|\frac{(x-1)^4}{x^3(x+2)}\right| + C'$$

$$\int \log(x^2+4)\,dx = \int (x)' \log(x^2+4)\,dx = x\log(x^2+4) - \int x \frac{2x}{x^2+4}\,dx$$

$$= x\log(x^2+4) - \int \frac{2x^2}{x^2+4}\,dx = x\log(x^2+4) - 2x + 8\int \frac{dx}{x^2+4} = x\log(x^2+4) - 2x + 4\arctan\left(\frac{x}{2}\right)$$

$$\int \frac{2x^2}{x^2+4}\,dx = 2\int \frac{(x^2+4)-4}{x^2+4}\,dx = 2\int \left(1 - \frac{4}{x^2+4}\right)dx$$

$$= 2x - 8\int \frac{1}{x^2+4}\,dx = 2x - 4\arctan\left(\frac{x}{2}\right)$$

$$8\int \frac{dx}{x^2+4} = \frac{8}{4}\int \frac{dx}{1+\left(\frac{x}{2}\right)^2} = 2\int \frac{2\left(\frac{x}{2}\right)'}{1+\left(\frac{x}{2}\right)^2}\,dx = 4\int \frac{1}{1+\left(\frac{x}{2}\right)^2}\underbrace{\left(\frac{x}{2}\right)'}_{\frac{dt}{dx}}\,dx$$

$t = \frac{x}{2}$ とおく

$$= 4\int \frac{1}{1+t^2}\underbrace{\frac{dt}{dx}\cdot\frac{dx}{dt}}_{=1}\,dt = 4\int \frac{dt}{1+t^2}$$

$$= 4\arctan t = 4\arctan\left(\frac{x}{2}\right)$$

$$f(x) = \frac{n\text{次多項式}}{m\text{次多項式}} = n-m\text{次多項式} + \frac{m-1\text{次以下の多項式}}{m\text{次多項式}}$$

$$f(x) = \underset{\substack{\uparrow \\ \text{多項式}}}{P(x)} + \sum \frac{C}{(x+a)^m} + \sum \frac{Ax+B}{(\cancel{x^2+bx+c})^m}$$

$$(x-a)^2+b^2$$

$$\int f(x)dx = \underbrace{\int P(x)\,dx}_{\text{積分可能}} + \sum C \int \frac{1}{(x+a)^n} dx + \sum \int \frac{Ax+B}{((x-a)^2+b^2)^m} dx$$

$$\int \frac{1}{(x+a)^n} dx = \frac{-1}{m-1} \frac{1}{(x+a)^{m-1}} + C$$

$$\int \frac{Ax+B}{((x-a)^2+b^2)^m} dx = \int \frac{A(x-a)+aA+B}{(\quad " \quad)^m} dx = \frac{A}{2}\int \frac{2(x-a)}{((x-a)^2+b^2)^m} dx$$

$$+ (aA+B)\int \frac{1}{((x-a)^2+b^2)^m} dx$$

$$I_m = \int \frac{2(x-a)}{((x-a)^2+b^2)^m} dx$$

$$= \int \frac{((x-a)^2+b^2)'}{((x-a)^2+b^2)^m} dx = \begin{cases} \dfrac{-1}{m-1} \dfrac{1}{((x-a)^2+b^2)^{m-1}} + C & (m>1) \\ \log|(x-a)^2+b^2| + C & (m=1) \end{cases}$$

$$J_m = \int \frac{1}{((x-a)^2+b^2)^m} dx$$

$\underline{m=1}$

$$J_1 = \int \frac{1}{(x-a)^2+b^2} dx = \frac{1}{b}\int \frac{\left(\frac{x-a}{b}\right)'}{1+\left(\frac{x-a}{b}\right)^2} dx$$

$$= \frac{1}{b}\arctan\left(\frac{x-a}{b}\right) + C$$

$m \geq 2$

$$J_m = \int \frac{dx}{((x-a)^2+b^2)^m} = \frac{1}{b^{2m-1}} \int \frac{\left(\frac{x-a}{b}\right)'}{\left(1+\left(\frac{x-a}{b}\right)^2\right)^m} dx$$

$$= \frac{1}{b^{2m-1}} \int \frac{dt}{(1+t^2)^m} \qquad \left(t = \frac{x-a}{b} \text{ とおく} \right)$$

よって $\Rightarrow$ $J_m = \dfrac{1}{b^{2m-1}} \displaystyle\int \frac{dt}{(1+t^2)^m}$ と書ける。

J_m と J_{m-1} の関係式をだす。

積分を変形して $m-1$ がでるようにする。

計算する。

$$J_m = \frac{1}{b^{2m-1}} \int \frac{(1+t^2)-t^2}{(1+t^2)^m} dt = \frac{1}{b^{2m-1}} \int \frac{dt}{(1+t^2)^{m-1}} - \frac{1}{b^{2m-1}} \int \frac{t^2}{(1+t^2)^m} dt$$

$$= \frac{1}{b^2} \overbrace{\boxed{\frac{1}{b^{2(m-1)-1}} \int \frac{dt}{(1+t^2)^{m-1}}}}^{J_{m-1}} - \frac{1}{b^{2m-1}} \int \frac{t^2}{(1+t^2)} dt$$

$$= \frac{1}{b^2} J_{m-1} - \frac{1}{b^{2m-1}} \int \frac{t^2}{(1+t^2)^m} dt \quad //$$

ここで

$$\int \frac{t^2}{(1+t^2)^m} dt = \frac{-1}{2(m-1)} \int t \left(\frac{1}{(1+t^2)^{m-1}} \right)' dt$$

$$\cancel{= \frac{(-1)}{2(m-1)} \int \frac{1}{(1+t^2)^{m-1}} dt + \frac{-1}{2(m-1)} t \frac{1}{(1+t^2)^{m-1}}}$$

$$= \frac{-1}{2(m-1)} \frac{t}{(1+t^2)^{m-1}} + \frac{+1}{2(m-1)} \int \frac{dt}{(1+t^2)^{m-1}}$$

もどして。

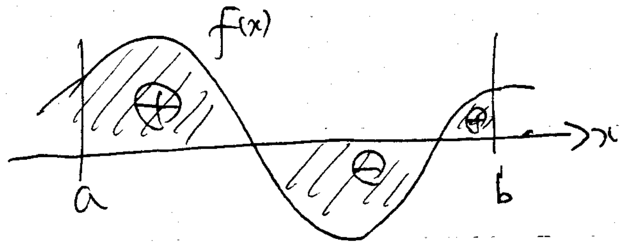
$$J_m = \cancel{\frac{1}{b^2} J_{m-1} - \frac{1}{b^{2m-1}} \frac{+1}{2(m-1)} \int \frac{dt}{(1+t^2)^{m-1}} \frac{1}{b^{2m-1}} \frac{1}{2(m-1)} \frac{t}{(1+t^2)^{m-1}}}$$

$$= \frac{1}{b^2} J_{m-1} - \frac{1}{2(m-1) b^2} \underbrace{\boxed{\frac{1}{b^{2(m-1)-1}} \int \frac{dt}{(1+t^2)^{m-1}}}}_{= J_{m-1}} + \frac{1}{2(m-1) b^{2m-1}} \frac{t}{(1+t^2)^{m-1}}$$

$$= \left(1 - \frac{1}{2(m-1)}\right) \frac{1}{b^2} J_{m-1} + \frac{1}{2(m-1) b^{2m-1}} \frac{t}{(1+t^2)^{m-1}}$$ 変数をもどして。

$$= \left(\frac{2m-3}{2m-2}\right) \frac{1}{b^2} J_{m-1} + \frac{1}{2(m-1)} \frac{1}{b^2} \frac{(x-a)}{((x-a)^2+b^2)^{m-1}} \quad //$$

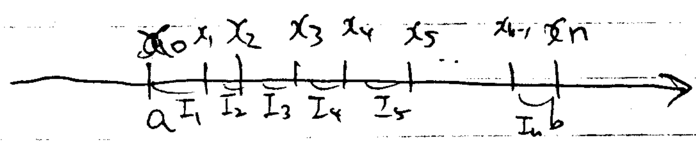
定積分



符号付きの面積. S

区間 $a \leq x \leq b$ において
x軸と $y=f(x)$ で囲まれた
領域の符号付きの面積 $= S$.

区間 $a \leq x \leq b$ を n個に区間に分ける.

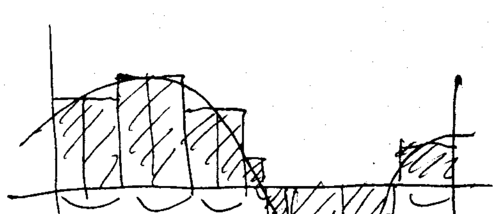


分割. $a=x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$.

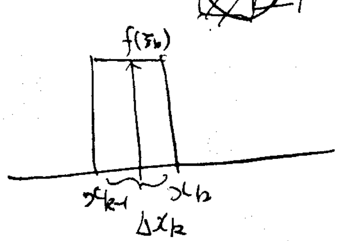
区間. $I_k = \{ x \mid x_{k-1} \leq x \leq x_k \}$ $k=1,2,\cdots,n$.

区間内のある点 $\xi_k \in I_k$

I_k 区間の幅 $\Delta x_k = x_k - x_{k-1}$



$$S_n = \sum_{k=1}^{n} f(\xi_k) \Delta x_k.$$



区間 $a \leq x \leq b$ の分割を増やす $n \to \infty$.
このときの面積 S'_n の極限 S を考える.

$\boxed{\text{定義}}$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(\xi_k)\Delta x_k .$$

関数 $f(x)$ の a から b までの定積分とよぶ.

極限 S が存在するとき $f(x)$ は積分可能であるという.

$\boxed{\text{例}}$

$$\int_a^b c\,dx \quad (c:\text{定数})$$

$$\int_a^b c\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(\xi_k)\Delta x_k = \lim_{n\to\infty}\sum_{k=1}^{n} c\,(x_k - x_{k-1})$$

$$= \lim_{n\to\infty} c\left((x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \cdots + (x_1 - x_0)\right)$$

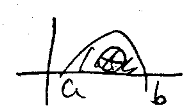
$$= \lim_{n\to\infty} c\,(x_n - x_0) = \lim_{n\to\infty} c\,(b-a) = c\,(b-a)$$

$\boxed{\text{定理}}$ (定積分の性質)

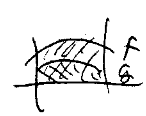
— $\displaystyle\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

— $\displaystyle\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx$.

— $f(x) \geq 0 \ (a \leq x \leq b)$ のとき $\displaystyle\int_a^b f(x)\,dx \geq 0$



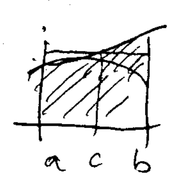
— $f(x) \geq g(x) \ (a \leq x \leq b)$ のとき $\displaystyle\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$



— $\displaystyle\int_a^b f(x)\,dx = f(c)(b-a) \quad a < c < b$

$\exists c \in [a, b)$

をみたす c が存在する.



(つづく)

(つづき)

— $a < c < b$ に対して、

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

— $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

— $$\int_a^a f(x)\,dx = 0.$$

aからb

$$\sum_{k=1}^{n} f(\xi_k)\Delta x_k$$

$$= \sum f(\xi)\Delta x_k + \sum f(\xi)$$ [illegible]

aからc　　cから[illegible]

$$\sum f(\xi_k)(x_k - x_{k-1})$$

↓

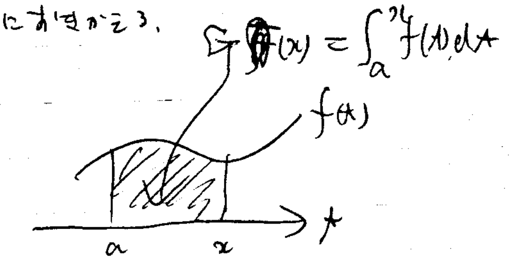
$$= -\sum f(\xi_k)(x_{k-1} - x_k)$$

定積分と不定積分

定積分 $\int_a^b f(x)\,dx$ と不定積分 $\int f(x)\,dx$ の関係を調べる。

$$\int_a^b \underbrace{f(x)\,dx}_{\text{変数}x\text{は何でもよい}} = \int_a^b f(t)\,dt$$

tにおきかえる。



範囲を $[a,b]$ から $[a,x]$ とする。

このとき

$$G(x) = \int_a^x f(t)\,dt$$

とおく。

$G(x)$ と $G(x+\Delta x)$ を考える。

Δx は微小量とする $\Delta x \sim 0$.

$$G(x+\Delta x) = \int_a^{x+\Delta x} f(t)\,dt = \int_a^x f(t)\,dt + \int_x^{x+\Delta x} f(t)\,dt$$

$$= G(x) + \int_x^{x+\Delta x} f(t)\,dt.$$

よって $G(x+\Delta x) - G(x) = \int_x^{x+\Delta x} f(t)\,dt$.

前述の定理より $x < \xi < x+\Delta x$

$$\int_x^{x+\Delta x} f(t)\,dt = f(\xi)(x+\Delta x - x)$$

$$= f(\xi)\Delta x.$$

となるので

$$\frac{G(x+\Delta x) - G(x)}{\Delta x} = f(\xi).$$

$\Delta x \to 0$ のとき $\xi \to x$ であるから

$$\frac{dG}{dx}(x) = f(x)$$

これはすなわち

$G(x)$ は $f(x)$ の原始関数である、ことを意味する

~~すなわち~~ $f(x)$ の原始関数を

よって $F(x) = \int f(x)\,dx$ とおくと

任意定数の不定性を考慮すると $G(x) = \int_a^x f(t)\,dt = F(x) + C$ が成り立つ.

C を定める。 $x = a$ のとき

$$0 = \int_a^a f(t)\,dt = F(a) + C$$

$$\to C = -F(a)$$

~~$x = b$ のとき~~ よって $C = -F(a)$ を得る.

以上より

$$\int_a^x f(t)\,dt = F(x) - F(a).$$

x を b に置きかえて, t を x にもどして.

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)} \to [F(x)]_a^b = F(x)\Big|_a^b$$

ただし $F(x) = \int f(x)\,dx$

これは

$$\boxed{\begin{aligned}\int_a^b f(x)\,dx &= [F(x)]_a^b \\ &= F(x)\Big|_a^b\end{aligned}}$$

とも書く.

定積分の計算

$F(x) = \int f(x)\,dx$ とする

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

例

— $f(x) = \alpha$ ← 定数

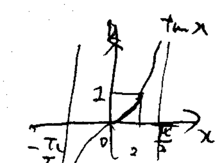
$$\int_a^b f(x)\,dx = \int_a^b \alpha\,dx = \alpha\int_a^b dx = \alpha\cdot[x]_a^b = \alpha(b-a)$$

— $f(x) = x$.

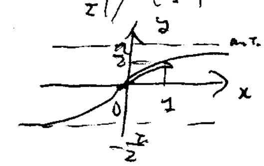
$$\int_a^b f(x)\,dx = \int_a^b x\,dx = \left[\frac{x^2}{2}\right]_a^b = \frac{b^2}{2} - \frac{a^2}{2} = \frac{1}{2}(b^2-a^2)$$
$$= \frac{1}{2}(b-a)(b+a)$$

— $f(x) = \cos x$.

$$\int_0^{\frac{\pi}{2}} \cos x\,dx = [\sin x]_0^{\frac{\pi}{2}} = \sin\frac{\pi}{2} - \sin 0$$
$$= 1 - 0 = 1$$



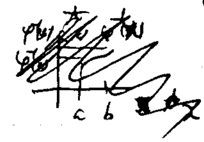
— $$\int_0^1 \frac{dx}{1+x^2} = [\arctan x]_0^1$$
$$= \arctan 1 - \arctan 0$$
$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

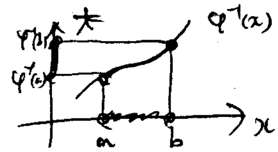


置換積分

$$\int_a^b f(x)\,dx = \int_{\varphi^{-1}(a)}^{\varphi^{-1}(b)} f(\varphi(t))\,\varphi'(t)\,dt = \int_{\varphi^{-1}(a)}^{\varphi^{-1}(b)} f(\varphi(t))\,\frac{dx}{dt}\,dt.$$

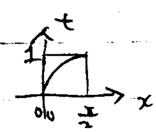
$$\begin{cases} x = \varphi(t) \\ t = \varphi^{-1}(x). \end{cases}$$





例

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin^2 x}\,dx.$$

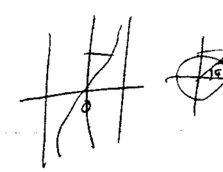


$t = \sin x$ とおく.

$$\frac{dt}{dx} = \cos x.$$

x	$0 \to \frac{\pi}{2}$
t	$0 \to 1$

$$I = \int_0^1 \frac{\frac{dt}{dx}}{1+t^2}\cdot\underbrace{\frac{dx}{dt}}_{1}\,dt. = \int_0^1 \frac{dt}{1+t^2}$$



$$= \Big[\arctan x\Big]_0^1 = \arctan 1 - \arctan 0$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4} //.$$

部分積分

$$\int_a^b f'(x)g(x)\,dx = \Big[f(x)g(x)\Big]_a^b - \int_a^b f(x)g'(x)\,dx$$

例

$$\int_0^\pi x\sin x\,dx = \int_0^\pi x(-\cos x)'\,dx$$

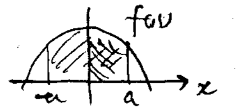
$$= \Big[-x\cos x\Big]_0^\pi - \int_0^\pi 1\,(-\cos x)\,dx$$

$$= \Big[-x\cos x\Big]_0^\pi + \int_0^\pi \cos x\,dx = \Big[-x\cos x + \sin x\Big]_0^\pi$$

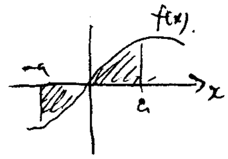
$$= (-\pi\underbrace{\cos\pi}_{-1} + \underbrace{\sin\pi}_{0}) - (-0 + \sin 0) = +\pi //.$$

その他の性質

$f(x)$ が偶関数. $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$



$f(x)$ が奇関数. $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$



$m, n \in \mathbb{N}$.

$$\int_0^{2\pi} \sin mx\,dx = 0$$

$$\int_0^{2\pi} \cos nx\,dx = 0.$$

$$\int_0^{2\pi} \sin mx \sin nx\,dx = \pi\delta_{n,m}$$

$$\int_0^{2\pi} \sin nx \cos mx\,dx = 0$$

$$\int_0^{2\pi} \cos nx \cos mx\,dx = \pi\delta_{n,m}.$$

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$$\int_0^{2\pi} \sin mx\,dx = \left[-\frac{1}{m}\cos mx\right]_0^{2\pi} = -\frac{1}{m}(\underbrace{\cos 2m\pi - \cos 0}_{\cos 0}) = 0.$$

$$\int_0^{2\pi} \cos nx\,dx = \left[\frac{1}{n}\sin nx\right]_0^{2\pi} = \frac{1}{n}(\underbrace{\sin 2n\pi}_{\sin 0} - \underbrace{\sin 0}_{0}) = 0.$$

$$\int_0^{2\pi} \sin mx \sin nx\,dx = -\frac{1}{2}\int_0^{2\pi}(\cos(m+n)x - \cos(m-n)x)\,dx = -\frac{1}{2}\int_0^{2\pi}\cos(m+n)x\,dx + \frac{1}{2}\int_0^{2\pi}\cos(m-n)x\,dx = 0$$

$$\int$$

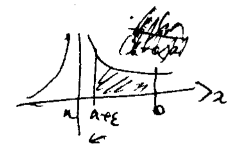
広義積分

連続な有限区間での~~定積~~積分 $\xrightarrow{\text{拡張}}$

不~~継~~連続点を含む区間での積分

~~無~~限区間での積分

= 広義積分 (improper integral)



$\boxed{\text{定義}}$ (不連続点を含む区間での広義積分)

関数 $f(x)$ が区間 $a < x \leq b$ で連続なとき，

もし ~~積分~~ 極限

$$\lim_{\varepsilon \to +0} \int_{a+\varepsilon}^{b} f(x)\,dx$$

が存在するならば，$f(x)$ は $a \leq x \leq b$ で積分可能であるという。

この極限値を

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to +0} \int_{a+\varepsilon}^{b} f(x)\,dx$$

で表す。

~~定義~~ 同様に

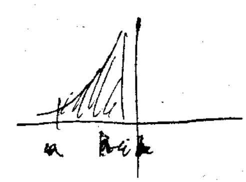
関数 $f(x)$ が区間 $a \leq x < b$ で連続で，極限

$$\lim_{\varepsilon \to +0} \int_a^{b-\varepsilon} f(x)\,dx$$

が存在するならば，

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to +0} \int_a^{b-\varepsilon} f(x)\,dx$$

と表す。



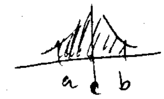
今，$f(x)$ が区間 $a \leq x \leq b$ 内の1点 c で不連続で，他の点では連続なとき，

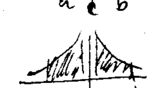
$$\int_a^b f(x)\,dx = \lim_{\varepsilon_1 \to +0} \int_a^{c-\varepsilon_1} f(x)\,dx + \lim_{\varepsilon_2 \to +0} \int_{c+\varepsilon_2}^{b} f(x)\,dx$$

別の変数

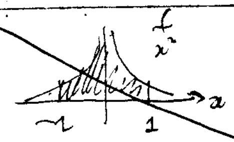
と定義する

以上の極限が存在するとき，広義積分は収束するという。





$\boxed{例}$ $\displaystyle\int_{-1}^{1}\frac{1}{x^2}dx$



$$\int_0^1\frac{dx}{x^2}=\lim_{\varepsilon\to+0}\int_{0+\varepsilon}^{1}\frac{dx}{x^2}=\lim_{\varepsilon\to+0}\left[-\frac{1}{x}\right]_{\varepsilon}^{1}$$

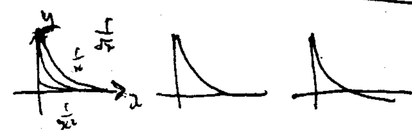
$$=\int_{-1}^{0}\frac{dx}{x^2}+\int_0^1\frac{dx}{x^2}\qquad =\lim_{\varepsilon\to+0}\left(-\frac{1}{1}+\frac{1}{\varepsilon}\right)=\infty$$

$$=\infty+\infty=\infty$$

$$\int_{-1}^{0}\frac{dx}{x^2}=\lim_{\varepsilon\to+0}\int_{-1}^{-\varepsilon}\frac{dx}{x^2}=\lim_{\varepsilon\to+0}\left[-\frac{1}{x}\right]_{-1}^{-\varepsilon}$$

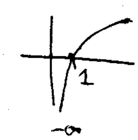
$$=\lim_{\varepsilon\to+0}\left(-\frac{1}{-\varepsilon}+\frac{1}{-1}\right)=+\infty$$

$\boxed{例}$ $\displaystyle\int_0^1\frac{dx}{\sqrt{x}},\ \int_0^1\frac{dx}{x},\ \int_0^1\frac{dx}{x^2}$



$$\int_0^1\frac{dx}{\sqrt{x}}=\lim_{\varepsilon\to+0}\int_{0+\varepsilon}^{1}\frac{dx}{\sqrt{x}}=\lim_{\varepsilon\to+0}\left[2\sqrt{x}\right]_{\varepsilon}^{1}=\lim_{\varepsilon\to+0}\left(2\sqrt{1}-2\sqrt{\varepsilon}\right)=2\ //$$

$$\int_0^1\frac{dx}{x}=\lim_{\varepsilon\to+0}\int_{0+\varepsilon}^{1}\frac{dx}{x}=\lim_{\varepsilon\to+0}\left[\log x\right]_{\varepsilon}^{1}=\lim_{\varepsilon\to+0}(\underbrace{\log 1}_{0}-\log\varepsilon)$$

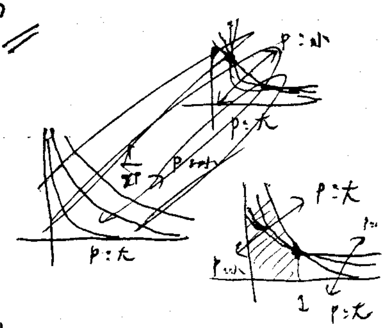


$$=-\lim_{\varepsilon\to+0}\log\varepsilon=+\infty\ //$$

$$\int_0^1\frac{dx}{x^2}=\lim_{\varepsilon\to+0}\int_{0+\varepsilon}^{1}\frac{dx}{x^2}=\lim_{\varepsilon\to+0}\left[-\frac{1}{x}\right]_{\varepsilon}^{1}=\lim_{\varepsilon\to+0}\left[-1+\frac{1}{\varepsilon}\right]=+\infty\ //$$

$\boxed{定理}$

$$\int_0^1\frac{dx}{x^p}=\begin{cases}\dfrac{1}{1-p}\in 有限 & (0<p<1)\\ +\infty & (p\geq 1)\end{cases}$$
$(p>0)$



(証明) $0<p<1$

$$\int_0^1\frac{dx}{x^p}=\lim_{\varepsilon\to+0}\int_{\varepsilon}^{1}\frac{dx}{x^p}=\lim_{\varepsilon\to+0}\left[\frac{1}{1-p}x^{1-p}\right]_{\varepsilon}^{1}=\lim_{\varepsilon\to+0}\left(\frac{1}{1-p}-\frac{\varepsilon^{\overset{0<1-p<1}{1-p}}}{1-p}\right)=\frac{1^{1-p}}{1-p}\ //$$

$p=1$

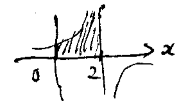
$$\int_0^1\frac{dx}{x^p}=\lim_{\varepsilon\to+0}\int_{\varepsilon}^{1}\frac{dx}{x}=\lim_{\varepsilon\to+0}\left[\log x\right]_{\varepsilon}^{1}=\lim_{\varepsilon\to+0}(\underbrace{\log 1}_{0}-\log\varepsilon)=+\infty\ //$$

$p>1$

$$\int_0^1\frac{dx}{x^p}=\lim_{\varepsilon\to+0}\int_{\varepsilon}^{1}\frac{dx}{x^p}=\lim_{\varepsilon\to+0}\left[\frac{-1}{p-1}\frac{1}{x^{p-1}}\right]_{\varepsilon}^{1}=\frac{-1}{p-1}+\frac{1}{p-1}\lim_{\varepsilon\to+0}\frac{1}{\underset{p-1>0}{\varepsilon^{p-1}}}=+\infty\ //$$

$\boxed{5(1)}$

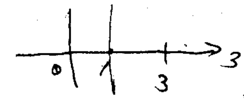
(1) $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{\varepsilon\to+0}\int_{0+\varepsilon}^1 \frac{dx}{\sqrt{x}} = \lim_{\varepsilon\to+0}\left[2\sqrt{x}\right]_\varepsilon^1 = \lim_{\varepsilon\to+0}\left(2-\sqrt{\varepsilon}\right) = 2$ //



(2) $\displaystyle\int_0^2 \frac{dx}{2-x} = \lim_{\varepsilon\to+0}\int_0^{2-\varepsilon}\frac{dx}{2-x} = \lim_{\varepsilon\to+0}\left[-\log(2-x)\right]_0^{2-\varepsilon}$

$\displaystyle = \lim_{\varepsilon\to+0}\left(-\log\varepsilon+\log 2\right) = +\infty.$ //

(3) $\cancel{\displaystyle\int_0^3 \frac{dx}{\sqrt[3]{x-1}}}$



定理（無限区間での広義積分）

関数 $f(x)$ が $x \geq a$ で連続で，極限

$$\lim_{b\to+\infty}\int_a^b f(x)\,dx$$

が存在するならば，この極限値を

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to+\infty}\int_a^b f(x)\,dx$$

と表す．同様に

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$$

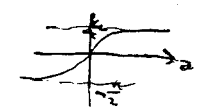
$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to-\infty}\lim_{b\to+\infty}\int_a^b f(x)\,dx$$

と定義する．これらの極限値が存在するとき，広義積分は収束するという．

例

$$\int_0^{\infty}\frac{dx}{x^2+4} = \lim_{b\to+\infty}\int_0^b\frac{dx}{x^2+4} = \lim_{b\to+\infty}\frac{1}{4}\int_0^b\frac{\left(\frac{x}{2}\right)'}{1+\left(\frac{x}{2}\right)^2}dx = \frac{1}{2}\lim_{b\to+\infty}\left[\arctan\frac{x}{2}\right]_0^b$$

$$= \frac{1}{2}\lim_{b\to+\infty}\left(\arctan\frac{b}{2} - \arctan 0\right) = \frac{1}{2}\left(\frac{\pi}{2}-0\right) = \frac{\pi}{4}$$



$$\int_{-\infty}^{0} e^{\alpha x}dx = \lim_{a\to-\infty}\int_a^0 e^{\alpha x}dx = \lim_{a\to-\infty}\left[\frac{1}{\alpha}e^{\alpha x}\right]_a^0 = \lim_{a\to-\infty}\frac{1}{\alpha}\left[1-e^{\alpha a}\right] = \frac{1-0}{\alpha} = \frac{1}{\alpha}$$

$(\alpha>0)$

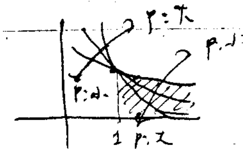
$$\int_1^{\infty}\frac{dx}{\sqrt{x}} = \lim_{b\to+\infty}\int_1^b\frac{dx}{\sqrt{x}} = \lim_{b\to+\infty}\left[2\sqrt{x}\right]_1^b = \lim_{b\to+\infty}(2\sqrt{b}-2) = \infty$$

$\boxed{\text{定理}}$

$$\int_1^{\infty}\frac{dx}{x^p} = \begin{cases} \frac{1}{p-1} & (p>1) \\ +\infty & (0<p\le 1) \end{cases}$$

$(p>0)$

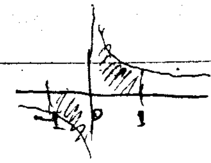
有限



(証明)

$$\int_1^{\infty}\frac{dx}{x^p} = \lim_{b\to+\infty}\int_1^b \frac{dx}{x^p} = \begin{cases} (p>1) \\ \lim_{b\to+\infty}\left[\frac{-1}{p-1}\frac{1}{x^{p-1}}\right]_1^b = \frac{-1}{p-1}\lim_{b\to+\infty}\left(\frac{1}{b^{p-1}}-1\right) = \frac{1}{p-1} \\ p=1 \\ \lim_{b\to+\infty}\left[\log x\right]_1^b = \lim_{b\to+\infty}\log b = +\infty \\ 0<p<1 \\ \lim_{b\to+\infty}\left[\frac{1}{1-p}x^{1-p}\right]_1^b = \frac{1}{1-p}\lim_{b\to+\infty}\left(b^{1-p}-1\right) = +\infty \end{cases}$$

コーシーの主値積分



例 $\displaystyle\int_{-1}^{1}\frac{dx}{x} = \int_{-1}^{0}\frac{dx}{x} + \int_{0}^{1}\frac{dx}{x} = \lim_{\varepsilon_1\to 0}\lim_{\varepsilon_2\to 0}\int_{-1}^{-\varepsilon_1}\frac{dx}{x} + \int_{\varepsilon_2}^{1}\frac{dx}{x}$

$\displaystyle = \lim_{\varepsilon_1\to 0}\left[\log|x|\right]_{-1}^{-\varepsilon_1} + \lim_{\varepsilon_2\to 0}\left[\log x\right]_{\varepsilon_2}^{1}$

$\displaystyle = \lim_{\varepsilon_1\to 0}\Big(\log\varepsilon_1 - \underbrace{\log 1}_{0}\Big) + \lim_{\varepsilon_2\to 0}\Big(\underbrace{\log 1}_{0} - \log\varepsilon_2\Big)$

$\displaystyle = \lim_{\varepsilon_1\to 0}\log\varepsilon_1 - \lim_{\varepsilon_2\to 0}\log\varepsilon_2 = -\infty + \infty$ 〜 不定.

$\boxed{\text{定義}}$ (コーシーの主値積分)

関数 $f(x)$ が $(a\le x\le b)$ から点 $x=c$ で不連続で其の他の点で連続なとき、積分

$$v.p.\int_{a}^{b} f(x)\,dx = \lim_{\varepsilon\to +0}\left(\int_{a}^{c-\varepsilon} f(x)\,dx + \int_{c+\varepsilon}^{b} f(x)\,dx\right)$$

を c におけるコーシーの主値積分 (Cauchy's principal value of integral) という.

同様に 関数 $f(x)$ ~~$(-\infty<x<\infty)$~~ が 区間 $-\infty<x<\infty$ で連続なとき 積分

$$v.p.\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to +\infty}\left(\int_{-a}^{a} f(x)\,dx\right) \qquad P\int_{a}^{b} f(x)\,dx$$

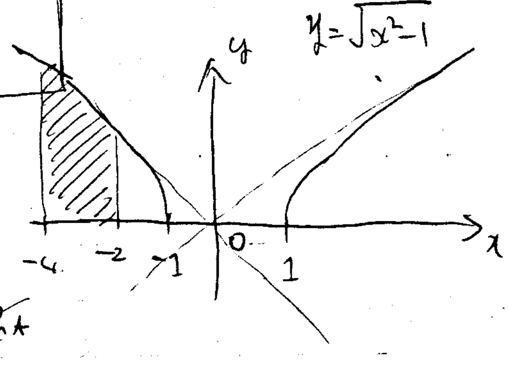
を ∞ におけるコーシーの主値積分という.

例 $\displaystyle\int_{-1}^{1}\frac{dx}{x} = \int_{-1}^{0}\frac{dx}{x} + \int_{0}\frac{dx}{x} = \lim_{\varepsilon\to +0}\left(\int_{-1}^{-\varepsilon}\frac{dx}{x} + \int_{\varepsilon}^{1}\frac{dx}{x}\right)$

$\displaystyle = \lim_{\varepsilon\to +0}\left(\left[\log|x|\right]_{-1}^{-\varepsilon} + \left[\log x\right]_{\varepsilon}^{1}\right) = \lim_{\varepsilon\to +0}\Big(\log\varepsilon - \underbrace{\log 1}_{0} + \underbrace{\log 1}_{0} - \log\varepsilon\Big)$

$\displaystyle = \lim_{\varepsilon\to +0}\left(\log\varepsilon - \log\varepsilon\right) = 0$

$$\boxed{I=\int_{-4}^{-2}\sqrt{x^2-1}\,dx=-\sqrt{3}+2\sqrt{15}-\frac{1}{2}\log\frac{2-\sqrt{3}}{4-\sqrt{15}}}$$



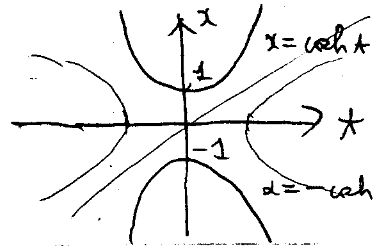
$x^2-1\geq 0$ より $|x|\geq 1$

積分区間 $-4\leq x\leq -2$ より $x\leq -1$ の範囲を考える.

$x=-\cosh t$ とおく.

$=-\frac{1}{2}(e^{t}+e^{-t})$

逆関数をとると



$t=\mathrm{arccosh}(-x)=\log\left(-x\pm\sqrt{x^2-1}\right)$ ← 2価関数. 枝が2本.

または 方程式 $x=\frac{-1}{2}(e^{t}+e^{-t})$ を tについて解く.

$-2x=e^{t}+e^{-t}$

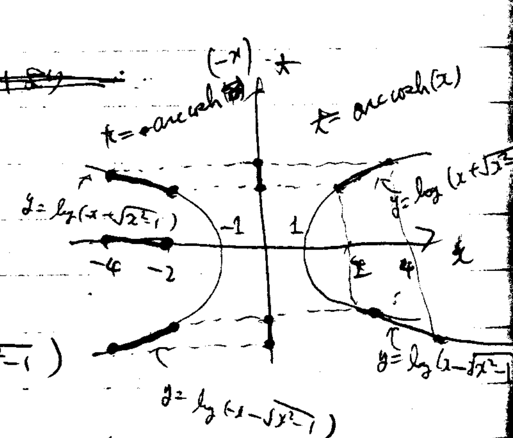
$-2xe^{t}=e^{2t}+1$

$e^{2t}+2xe^{t}+1=0$

$(e^{t}+x)^2=x^2-1$

$0\leq e^{t}=-x\pm\sqrt{x^2-1}$

~~$t=\log$~~ → $t=\log\left(-x\pm\sqrt{x^2-1}\right)$



x	$-4\to-2$
t	$\mathrm{arccosh}(4)\to\mathrm{arccosh}(2)$

$\mathrm{arccosh}(4)=\log(4\pm\sqrt{16-1})=\log(4\pm\sqrt{15})$

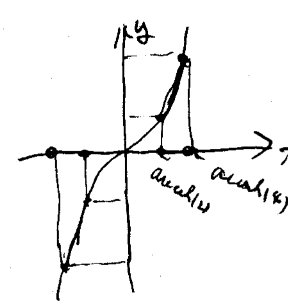
$\mathrm{arccosh}(2)=\log(2\pm\sqrt{4-1})=\log(2\pm\sqrt{3})$

$y=\sinh t=$

$$I=\int_{-4}^{-2}\sqrt{x^2-1}\,dx=\int_{\mathrm{arccosh}(4)}^{\mathrm{arccosh}(2)}\sqrt{\cosh^2 t-1}\,(-\sinh t)\,dt=-\int_{\mathrm{arccosh}(4)}^{\mathrm{arccosh}(2)}|\sinh t|\sinh t\,dt$$

$\sqrt{\cosh^2 t-1}=\sqrt{\sinh^2 t}=|\sinh t|$

$$=\int_{\mathrm{arccosh}(2)}^{\mathrm{arccosh}(4)}|\sinh t|\sinh t\,dt$$



⊕の方の枝を選ぶと $\mathrm{arccosh}(2)\leq t\leq\mathrm{arccosh}(4)$ このとき $\sinh t\geq 0$

⊖の方の枝を選ぶと $\mathrm{arccosh}(4)\leq t\leq\mathrm{arccosh}(2)$ このとき $\sinh t\leq 0$

⊕の枝. $I_+=\int_{\mathrm{arccosh}(2)}^{\mathrm{arccosh}(4)}(\sinh t)^2\,dt=\frac{1}{4}\int_{\bigcirc}^{\bigcirc}(e^{2t}+e^{-2t}-2)\,dt=\frac{1}{4}\left[\frac{e^{2t}}{2}-\frac{e^{-2t}}{2}-2t\right]_{\bigcirc}^{\bigcirc}$

⊖の枝 $I_-=\int_{\mathrm{arccosh}(4)}^{\mathrm{arccosh}(2)}(\sinh t)^2\,dt$

$$I_+ = \left[\frac{1}{8}e^{2t} - \frac{1}{8}e^{-2t} - \frac{t}{2}\right]_{\operatorname{arcosh}(2) = \log(2+\sqrt{3})}^{\operatorname{arcosh}(4) = \log(4+\sqrt{15})}$$

$$= \left[\frac{1}{4}\sinh(2t) - \frac{t}{2}\right]$$

$$= \frac{1}{4}\Big(\sinh(2\operatorname{arcosh}(4)) - \sinh(2\operatorname{arcosh}(2))\Big)$$

$$- \frac{1}{2}\Big(\underbrace{\operatorname{arcosh}(4) - \operatorname{arsinh}(2)}_{\log(4+\sqrt{15}) - \log(2+\sqrt{3}) \,=\, \log\frac{4+\sqrt{15}}{2+\sqrt{3}}}\Big)$$

$$\frac{(4+\sqrt{15})(4-\sqrt{15})}{(2+\sqrt{3})(4-\sqrt{15})} = \frac{1}{(2+\sqrt{3})(4-\sqrt{15})} = \frac{2-\sqrt{3}}{4-\sqrt{15}}$$

$$\sinh(2\operatorname{arcosh}(4)) = \frac{1}{2}\left(e^{2\log(4+\sqrt{15})} - e^{-2\log(4+\sqrt{15})}\right)$$

$$= \frac{1}{2}\left(e^{\log(4+\sqrt{15})^2} - e^{\log(4+\sqrt{15})^{-2}}\right)$$

$$= \frac{1}{2}\left((4+\sqrt{15})^2 - (4+\sqrt{15})^{-2}\right) = \frac{1}{2}\left((4+\sqrt{15})^2 - (4-\sqrt{15})^2\right)$$

$$= \frac{1}{2}(4+\cancel{\sqrt{15}}+4-\cancel{\sqrt{15}})(\cancel{4}+\sqrt{15}-\cancel{4}+\sqrt{15})$$

$$= \frac{1}{2}\times 8 \times 2\sqrt{15} = 8\sqrt{15}$$

$$\frac{1}{4+\sqrt{15}} = \frac{4-\sqrt{15}}{(4+\sqrt{15})(4-\sqrt{15})} = 4-\sqrt{15}$$

$$\frac{1}{2+\sqrt{3}} = 2-\sqrt{3}$$

$$\sinh(2\operatorname{arcosh}(4)) = 8\sqrt{15}$$

$$\sinh(2\operatorname{arcosh}(2)) = \cdots = \frac{1}{2}\left((2+\sqrt{3})^2 - (2-\sqrt{3})^2\right) = \frac{1}{2}(2+\cancel{\sqrt{3}}+2-\cancel{\sqrt{3}})(\cancel{2}+\sqrt{3}-\cancel{2}+\sqrt{3})$$

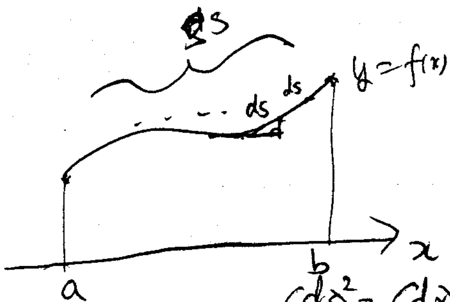
$$= \frac{1}{2}\times 4 \times 2\sqrt{3} = 4\sqrt{3}.$$

$$I_+ = \frac{1}{4}(8\sqrt{15} - 4\sqrt{3}) - \frac{1}{2}\log\frac{4+\sqrt{15}}{2+\sqrt{3}} = 2\sqrt{15} - \sqrt{3} - \frac{1}{2}\log\frac{4+\sqrt{15}}{2+\sqrt{3}}$$

$$I_- = \left[\frac{1}{4}\sinh(2t) - \frac{t}{2}\right]_{\operatorname{arcosh}(4) = \log(4-\sqrt{15}) \,=\, \log\frac{1}{4+\sqrt{15}} \,=\, -\log(4+\sqrt{15})}^{\operatorname{arcosh}(2) = \log(2-\sqrt{3}) \,=\, \log\frac{1}{2+\sqrt{3}} \,=\, -\log(2+\sqrt{3})}$$

$$I_- = -\frac{1}{4}\sinh(2\log(2+\sqrt{3})) - \frac{-1}{4}\sinh(2\log(4+\sqrt{15})) - \frac{-1}{2}\left(\log(2+\sqrt{3}) + \log(4+\sqrt{15})\right) = I_+$$

曲線の長さ



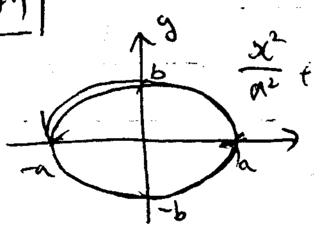
$$s = \int_0^{s} ds = \int \cancel{\sqrt{(dx)^2+(dy)^2}} = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$(ds)^2 = (dx)^2 + (dy)^2$$

$$ds = \sqrt{(dx)^2+(dy)^2}$$

$$= \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

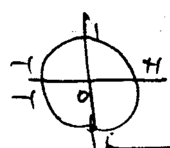
$\boxed{例}$



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad (a>0,\ b>0)$$

$$s = \int_{-1}^{1}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx + \int_{-1}^{1}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 4\int_0^1 \frac{dx}{\sqrt{1-x^2}} = 4\Big[\arcsin x\Big]_0^1$$



$$y = \pm\sqrt{1-x^2}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{1}{\sqrt{1-x^2}}$$

$$= 4(\underbrace{\arcsin 1}_{\frac{\pi}{2}} - \underbrace{\arcsin 0}_{0})$$

$$= 4\times\frac{\pi}{2} = 2\pi \ //$$

$$\frac{y^2}{b^2} = 1-\frac{x^2}{a^2}$$

$$y^2 = b^2\left(1-\frac{x^2}{a^2}\right)$$

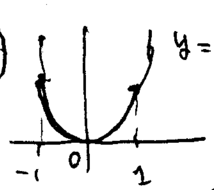
$$y = \pm b\sqrt{1-\frac{x^2}{a^2}}$$

$$\frac{dy}{dx} = \pm b\left(1-\frac{x^2}{a^2}\right)^{-\frac{1}{2}}\times\frac{1}{2}\times\left(-\frac{2x}{a^2}\right)$$

$$= \frac{\mp b\frac{x}{a^2}}{\sqrt{1-x^2/a^2}} \ //$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{b^2\frac{x^2}{a^4}}{(1-x^2/a^2)}} = \sqrt{\frac{1-\frac{x^2}{a^2}+\frac{b^2}{a^4}x^2}{1-x^2/a^2}} = \sqrt{\frac{1-\frac{a^2-b^2}{a^4}x^2}{1-x^2/a^2}}$$

例



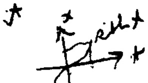
$$\frac{dy}{dx} = 2x$$

$$s = \int_{-1}^{1}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_{-1}^{1}\sqrt{1+4x^2}\, dx = 2\int_0^1\sqrt{1+(2x)^2}\, dx$$

$$\overset{2x\to x}{=} 2\int_0^2\sqrt{1+x^2}\,\frac{1}{2}dx = \int_0^2\sqrt{1+x^2}\, dx = \int_0^{\mathrm{arcsinh}(2)} \cosh t\,\frac{dx}{dt}\, dt$$

$$\frac{dx}{dt} = \cosh t \quad x = \sinh t$$

$$1+x^2 = 1+\sinh^2 t = \cosh^2 t$$



$$= \int_0^{\mathrm{arcsinh}(2)} \cosh^2 t\, dt$$

$$\begin{array}{c|c} x & 0 \to 2 \\ t & \mathrm{arcsinh}(0) \to \mathrm{arcsinh}\,2 \end{array}$$

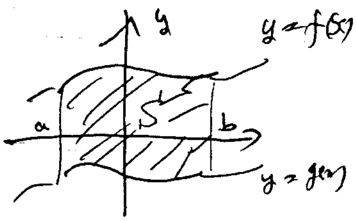
$$= \frac{1}{4}\int_0^{\sinh^{-1}(2)}\left(e^{2t}+e^{-2t}+2\right)dt = \left[\frac{e^{2t}}{4} - \frac{e^{-2t}}{8} + \frac{2t}{4}\right]_0^{\sinh^{-1}(2)} = \frac{(2+\sqrt{5})^2}{8} - \frac{(2+\sqrt{5})^{-2}}{8} + \frac{1}{2}\log(2+\sqrt{5})$$

$$\frac{1}{2+\sqrt{5}} = \frac{2-\sqrt{5}}{4-5} = -(2-\sqrt{5})$$

$$(2+\sqrt{5})^2 - (2-\sqrt{5})^2$$

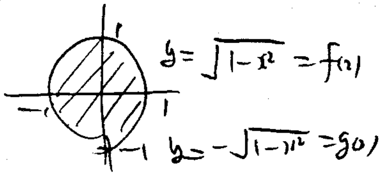
$$= 4+4\sqrt{5}+5 - 4 - 5 + 4\sqrt{5} = 8\sqrt{5}$$

図形の面積



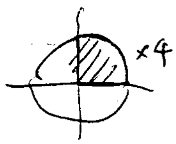
$$S = \int_a^b (f(x) - g(x))\,dx.$$

例



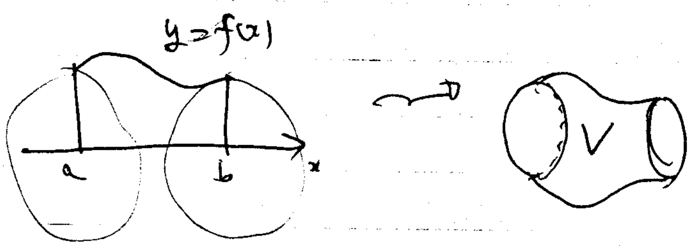
$$S = \int_{-1}^{1} (f(x) - g(x))\,dx = 2\int_{-1}^{1} \sqrt{1-x^2}\,dx = 4\int_0^1 \sqrt{1-x^2}\,dx$$

$$= 4 \times \frac{\pi}{4} = \pi$$



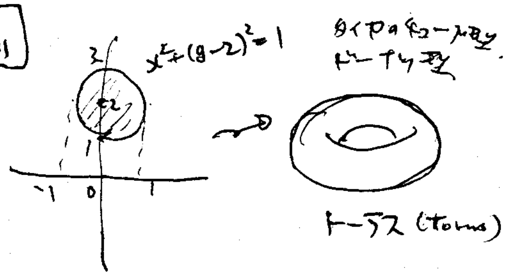
問 これを示せ.

回転体の体積



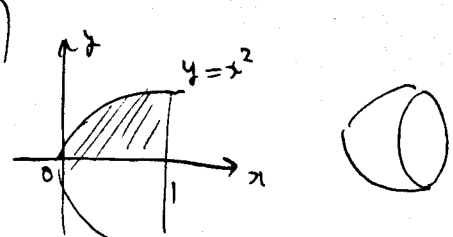
$$V = \pi \int_a^b (f(x))^2\,dx.$$

例



問 これを求めよ.

例



$$V = \pi \int_0^1 (x^2)^2\,dx = \pi \int_0^1 x^4\,dx$$

$$= \pi \left[\frac{x^5}{5}\right]_0^1 = \frac{\pi}{5}$$